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RIGHT-HAND SOLUTIONS OF THE DIFFERENTIAL EQUATIONS OF DYNAMICS FOR MECHANICAL SYSTEMS WITH SLIDING FRICTION†

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As a further development of Painlevé's theory [1], the existence, continuability and uniqueness of right-hand solutions of the differential equations of dynamics, and, under certain additional conditions, of the equations of motion of holonomic mechanical systems with sliding friction [2] are considered. In classical mechanics, acceleration is essentially defined as the right-hand derivative of velocity (see [3, 4]). Hence the most meaningful definition of the "solution of a differential equation" in problems of the dynamics of mechanical systems with sliding friction is that using the concept of right derivative [5].

From the mathematical point of view, right-hand solutions of the differential equations of motion for mechanical systems with sliding friction, in which the generalized friction forces are defined by Coulomb's law, must be considered in the problem of continuation to the right of local (classical) solutions, since generalized accelerations may experience discontinuities at points where the generalized velocities vanish (see [6]). Under special initial conditions, in which the friction forces at rest and in motion are the same, classical solutions may not exist at all, even locally.

1. EQUATIONS OF DYNAMICS

The equations of the dynamics of a mechanical system with sliding friction, may be written in terms of generalized coordinates $q = (q^1, \ldots, q^k)$ as

$$\sum_{i=1}^{k} a_{si}(t,q)\ddot{q}^i = Q_s^{T0}(t,q,\dot{q},\ddot{q}) + g_s(t,q,\dot{q}) + Q_s^A(t,q,\dot{q}), \quad s \in \mathcal{N}_0$$

$$\sum_{i=1}^{k} a_{si}(t,q)\ddot{q}^i = f_s(t,q^s,0)|N_s(t,q,\dot{q},\ddot{q})|\operatorname{sgn} Q_s^{T0}(t,q,\dot{q},\ddot{q}) + g_s(t,q,\dot{q}) + Q_s^A(t,q,\dot{q}), \quad s \in \mathcal{N} \setminus \mathcal{N}_0 \tag{1.1}$$

$$\sum_{i=1}^{k} a_{si}(t,q)\ddot{q}^i = f_s(t,q^s,0)|N_s(t,q,\dot{q},\ddot{q})|\operatorname{sgn} \dot{q}^s + g_s(t,q,\dot{q}) + Q_s^A(t,q,\dot{q}), \quad s \in (1,\ldots,k_*) \setminus \mathcal{N}$$

$$\sum_{i=1}^{k} a_{si}(t,q)\ddot{q}^i = g_s(t,q,\dot{q}) + Q_s^A(t,q,\dot{q}), \quad s = k_* + 1,\ldots,k$$

where $A(t,q) = [a_{ij}(t,q)]_1^k$ is the continuously differentiable, symmetric positive matrix of the coefficients of inertia, $Q^A(t,q,\dot{q})$, $g(t,q,\dot{q})$ are continuous vector-valued functions representing the active forces, generalized gyroscopic and other forces and terms, $Q_s^{T0}(t,q,\dot{q},\ddot{q})$ are the generalized forces of friction at relative rest, as expressed by the formulae

$$Q_s^{T0}(t,q,\dot{q},\ddot{q}) = \sum_{\substack{i=1 \\ i \neq s}}^{k} a_{si}(t,q)\ddot{q}^i - [g_s(t,q,\dot{q}) + Q_s^A(t,q,\dot{q})]_{\dot{q}^s=0}$$

$f_s(t, q^s, \dot{q}^s)$, $s = 1, \ldots, k_*$, $1 \leqslant k_* \leqslant k$ are the coefficients of friction (continuous functions),

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$|N_s(t, q, \dot{q}, \ddot{q})|$ are the absolute values of the normal reactions (at the points of contact of the bodies in friction), which are continuous together with their partial derivatives with respect to $\ddot{q}$ in the domain $\{(t, q, \dot{q}, \ddot{q}): |N_s(t, q, \dot{q}, \ddot{q})| \neq 0\}$, and

$$\mathcal{N} = \mathcal{N}(\dot{q}) \stackrel{\Delta}{=} \{s \in (1, \ldots, k_*): \dot{q}^s = 0\}$$

$$\mathcal{N}_0 = \mathcal{N}_0(t, q, \dot{q}, \ddot{q}) \stackrel{\Delta}{=} \{s \in \mathcal{N}: |Q_s^{T0}(t, q, \dot{q}, \ddot{q})| \leqslant f_s(t, q^s, 0)|N_s(t, q, \dot{q}, \ddot{q})|\}$$

If the inequality

$$f_s(t, q^s, 0)|\partial |N_s(t, q, \dot{q}, \ddot{q})|/\partial \ddot{q}^s| < a_{ss}(t, q) \tag{1.2}$$

holds for $s \in \mathcal{N}(\dot{q}) \setminus \mathcal{N}_0(t, q, \dot{q}, \ddot{q})$ and $(t, q, \dot{q}, \ddot{q})$ in the domain of definition $\Omega \times R^k$ such that $|N_s(t, q, \dot{q}, \ddot{q})| \neq 0$, then, as proved in [2], Eqs (1.1) are equivalent to the equations of motion of a mechanical system with sliding friction as described in [2], i.e. the solutions of these systems, understood in some sense or another (the classical solutions or right-hand solutions), are identical.

Let us introduce the following notation

$$\mathcal{N}_1 = \mathcal{N}_1(t, q, \dot{q}, \ddot{q}) \stackrel{\Delta}{=} \{s \in \mathcal{N}_0: |Q_s^{T0}(t, q, \dot{q}, \ddot{q})| = f_s(t, q^s, 0)|N_s(t, q, \dot{q}, \ddot{q})|\}$$

and $\{A(t, q)\}$ (k_*) denotes the submatrix of A obtained by deleting the first k_* rows and columns. Like $A(t, q)$, $[A(t, q)]$ (k_*) is positive-definite and therefore non-singular (see [7]). Let $[\tilde{a}_{k_*+i, k_*+j}]_1^{k-k_*}$ denote the inverse of $[A(t, q)]$ (k_*).

Solving the fourth group of equations (1.1) for $\ddot{q}^s$, $s = k_* + 1, \ldots, k$, we rewrite (1.1) in an equivalent semi-explicit form

$$\begin{aligned}
&\ddot{q}^s = 0, \quad s \in \mathcal{N}_0 \\
&\ddot{q}^s = a_{ss}^{-1}(t, q)[f_s(t, q^s, 0)|N_s(t, q, \dot{q}, \ddot{q})| \operatorname{sgn} Q_s^{T0}(t, q, \dot{q}, \ddot{q}) - Q_s^{T0}(t, q, \dot{q}, \ddot{q})], \quad s \in \mathcal{N} \setminus \mathcal{N}_0 \\
&\ddot{q}^s = a_{ss}^{-1}(t, q)[-f_s(t, q^s, 0)|N_s(t, q, \dot{q}, \ddot{q})| \operatorname{sgn} \dot{q}^s - Q_s^{T0}(t, q, \dot{q}, \ddot{q})], \quad s \in (1, \ldots, k_*) \setminus \mathcal{N} \\
&\ddot{q}^s = \sum_{i=k_*+1}^{k} \tilde{a}_{si}(t, q)[g_i(t, q, \dot{q}) + Q_i^A(t, q, \dot{q})] - \sum_{\nu=1}^{k_*} \sum_{i=k_*+1}^{k} \tilde{a}_{si}(t, q) a_{i\nu}(t, q) \ddot{q}^\nu, \quad s = k_* + 1, \ldots, k
\end{aligned} \tag{1.3}$$

2. THE SOLVABILITY OF THE EQUATIONS OF DYNAMICS FOR ACCELERATIONS

Let $F_s(t, q, \dot{q}, \ddot{q})$ $(s = 1, \ldots, k)$ denote the right-hand sides of Eqs (1.3). For $s = k_* + 1, \ldots, k$, the functions F_s do not depend on the variables $\ddot{q}^{k_*+1}, \ldots, \ddot{q}^k$. We may therefore put $\ddot{q}^s = F_s(t, q, \dot{q}, \ddot{q}^1, \ldots, \ddot{q}^{k_*})$ for $s = k_* + 1, \ldots, k$ and, accordingly, transfer from the first three groups of equations (1.3) to a new system of equations in $\ddot{q}^1, \ldots, \ddot{q}^k$. As a result we obtain

$$\ddot{q}^s = F_s(t, q, \dot{q}, \ddot{q}^1, \ldots, \ddot{q}^{k_*}, F_{k_*+1}, \ldots, F_k), \quad s = 1, \ldots, k_* \tag{2.1}$$

Determining the implicitly defined functions $\ddot{q}^s = G_s(t, q, \dot{q})$ $(s = 1, \ldots, k_*)$ from (2.1) (if they exist), we can obtain a vector-valued function $\ddot{q} = G(t, q, \dot{q})$ that satisfies (1.3) or, equivalently, (1.1), having put $G_s = F_s(t, q, \dot{q}, G_1, \ldots, G_{k_*})$ for $s = k_* + 1, \ldots, k$. Since F_s $(s = k_* + 1, \ldots, k)$ are jointly continuous in their arguments, the continuity properties of $G(t, q, \dot{q})$ will be determined by those of the functions $G_s(t, q, \dot{q})$ $(s = 1, \ldots, k_*)$.

Let $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ be a point such that $|N_s(t_0, q_0, \dot{q}_0, \ddot{q}_0)| \neq 0$ $(s \in (1, \ldots, k_*) \setminus (\mathcal{N}_0 \setminus \mathcal{N}_1))$ and the system of equations (1.1) is satisfied. Then for any $s \in \mathcal{N}_1 = \mathcal{N}_1(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ we have $Q_s^{T0}(t_0, q_0, \dot{q}_0, \ddot{q}_0) \neq 0$. We define a set of indices

$$\mathcal{N}_2 = \mathcal{N}_2(t, q, \dot{q}, G, t_0, q_0, \dot{q}_0, \ddot{q}_0) \stackrel{\Delta}{=} (\mathcal{N}_0(t, q, \dot{q}, G) \setminus \mathcal{N}_1(t, q, \dot{q}, G)) \cap \mathcal{N}_1(t_0, q_0, \dot{q}_0, \ddot{q}_0)$$

and a set of directions in space

$$\Gamma(t, q, \dot{q}, \ddot{q}) = \{\upsilon \in R^k: \upsilon_s = 0, s \in \mathcal{N}_0 \setminus \mathcal{N}_1; \tag{2.2}$$

$$-\operatorname{sgn} Q_s^{T0}(t,q,\dot q,\ddot q)\upsilon_s \geqslant 0, \quad s \in \mathcal{N}_1 \cup (\mathcal{N} \setminus \mathcal{N}_0)\}$$

and we put $\Gamma_0 = \Gamma(t_0, q_0, \dot q_0, \ddot q_0)$. For arbitrary $\delta > 0$, we put

$$S_\delta = S_\delta(\ddot q_0) \triangleq \{\ddot q : \|\ddot q - \ddot q_0\| < \delta\}$$

$$D_\delta = D_\delta(t_0, q_0, \ddot q_0) \triangleq \{(t,q,\dot q) : t_0 \leqslant t < \delta$$
$$\|q - q_0\| < \delta, \ \|\dot q - \dot q_0\| < \delta, \ \dot q \in \Gamma_0\}$$

where $\|\cdot\|$ is some norm in the space R^k.

Consider the following system of inequalities at the point $(t_0, q_0, \dot q_0, \ddot q_0)$, for all $s \in (1, \ldots, k_*) \setminus (\mathcal{N}_0 \setminus \mathcal{N}_1)$

$$|a_{s\nu}\gamma_{s\nu} - f_s \frac{\partial |N_s|}{\partial \ddot q^\nu}\Delta_s + \sum_{j=k_*+1}^{k}\left(a_{sj} - f_s \frac{\partial |N_s|}{\partial \ddot q^j}\Delta_s\right)\sum_{i=k_*+1}^{k} \tilde a_{ji} a_{i\nu}| < \frac{a_{ss}e_{s\nu}}{k_*(n-1)} \tag{2.3}$$

$$\gamma_{s\nu} = \begin{cases} 0, & s = \nu; \\ 1, & s \neq \nu; \end{cases} \quad \Delta_s = \begin{cases} -\operatorname{sgn} Q_s^{T0}(t,q,\dot q,\ddot q) & s \in \mathcal{N} \setminus (\mathcal{N}_0 \setminus \mathcal{N}_1) \\ .\operatorname{sgn} \dot q^s, & s \in (1,\ldots,k_*) \setminus \mathcal{N} \end{cases}$$

if $\nu = 1, \ldots, k_*$, where $e_{s\nu}$ is the identity of the appropriate dimension (so that the dimensions of the left- and right-hand sides of the inequality are the same) and n is the number of indices in N_1 when $n > 2$ (if $n = 1, 2$ or $\mathcal{N}_1 = \phi$, we put $n = 2$). Inequalities (2.3) will always be valid for sufficiently small coefficients of friction f_s and the off-diagonal elements $a_{s\nu}$ ($s, \nu = 1, \ldots, k_*, s \neq \nu$) of A.

If all the functions involved in the terms F_s on the right of (2.1) satisfy a Lipschitz condition as functions of $(\ddot q^1, \ldots, \ddot q^{k_*})$, with the same constant $0 \leqslant L < 1$, then, by continuity with respect to $(\ddot q^1, \ldots, \ddot q^{k_*})$ the same will be true of the functions F_s themselves. Accordingly, one can show that the validity of inequalities (2.3) at the point $(t_0, q_0, \dot q_0, \ddot q_0)$ is a sufficient condition for each function F_s on the right of (2.1), as a function of $(\ddot q^1, \ldots, \ddot q^{k_*})$ (and for all the functions involved in the terms F_s, as functions of $(t_0, q_0, \dot q_0, \ddot q_0)$), to be contractive relative to the norm $\|\ddot q\|_1 = \max_{1 \leqslant i \leqslant k_*} |\ddot q^i\|$ in some neighbourhood of $(t_0, q_0, \dot q_0, \ddot q_0)$ for any fixed $(t, q, \dot q) \in D_\delta$. Then, obviously, the vector-valued function $F = (F_1, \ldots, F_{k_*})$ will be contractive as a function of $(\ddot q^1, \ldots, \ddot q^{k_*})$, if the space R^{k_*} of values of that function is considered with a norm $\|\cdot\|_1$. Now, using the equality $\ddot q_0 = F(t_0, q_0, \dot q_0, \ddot q_0)$ and the principle of contractive mappings, we conclude that arbitrarily small numbers $\delta_0 > 0$, $\delta_1 > 0$ exist such that there is a unique function $G(t, q, \dot q)$, defined on D_{δ_0} with values in S_{δ_1}, that is a solution of Eqs (1.3).

Thus, for $(t, q, \dot q, \ddot q) \in D_{\delta_0} \times S_{\delta_1}$ Eqs (1.3) are equivalent to the equation $\ddot q = G(t, q, \dot q)$. Moreover, the numbers δ_0 and δ_1 may be chosen so that

$$\operatorname{sgn} Q_s^{T0}(t,q,\dot q,\ddot q) = \operatorname{sgn} Q_s^{T0}(t_0,q_0,\dot q_0,\ddot q_0) \tag{2.4}$$

for all $s \in \mathcal{N}_1(t_0, q_0, \dot q_0, \ddot q_0)$, $(t, q, \dot q, \ddot q) \in D_{\delta_0} \times S_{\delta_1}$.

We will list a few important properties of G.

1. G is continuous at every point $(t, q, \dot q) \in D_{\delta_0}$ for which $\mathcal{N}_2(t, q, \dot q, G, t_0, q_0, \dot q_0, \ddot q_0) \neq \phi$.

2. At every point $(t, q, \dot q) \in D_{\delta_0}$ the function G has a finite number of limit values not exceeding 2^n, where n is the number of elements in the set $\mathcal{N}_1(t_0, q_0, \dot q_0, \ddot q_0)$.

3. If $\bar G$ is a limit value of G at the point $(t, q, \dot q) \in D_{\delta_0}$, different from $G(t, q, \dot q)$, then $\mathcal{N}_2 \supset I^+ \neq \phi$ (an index $s_0 \in I^+$ exists) and

$$\sum_{s \in I^-} |\bar G_s| < \sum_{s \in I^+} |\bar G_s| \tag{2.5}$$

where

$$I^+ = I^+(t,q,\dot q,\bar G) \triangleq \{s \in \mathcal{N}_2 : \bar G_s \neq 0, \ \operatorname{sgn} \bar G_s = Q_s^{T0}(t,q,\dot q,\bar G)\}$$

$$I^- = I^-(t,q,\dot q,\bar G) \triangleq \{s \in \mathcal{N}_2 : \bar G_s \neq 0, \ \operatorname{sgn} \bar G_s = -\operatorname{sgn} Q_s^{T0}(t,q,\dot q,\bar G)\}$$

and the left-hand side of (2.5) is put equal to zero if $I^- = \phi$.

The following proposition is a direct corollary of property 3.

Proposition 2.1. For each point of discontinuity $(t, q, \dot{q}) \in D_{\delta 0}$ of G, a vector $a \in R^k$ exists with the following properties.

1. $\langle a, G(t, q, \dot{q})\rangle = \langle a, \dot{q}\rangle = 0$;
2. $\langle a, \upsilon\rangle \geqslant 0$ for all $\upsilon \in \Gamma_0$;
3. $\langle a, \bar{G}(t, q, \dot{q})\rangle < 0$ for all limit values $\bar{G}(t, q, \dot{q})$ of G at the point $(t, q, \dot{q})$ that differ from $G(t, q, \dot{q})$,

where $\langle \cdot , \cdot \rangle$ is the scalar product.

Indeed, if $(t, q, \dot{q})$ is a point of discontinuity of G, it follows from the definition of the sets $\mathcal{N}_2$ and Γ_0 and from inequalities (2.5) and (2.4) that these properties hold for the vector a whose components are $-\operatorname{sgn} Q_s^{T0}(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ for $s \in \mathcal{N}_2(t, q, \dot{q}, \ddot{q}, G, t_0, q_0, \dot{q}_0, \ddot{q}_0)$, its other components being zero.

3. A THEOREM OF THE EXISTENCE OF A SOLUTION

Given a continuous function q: $[t_0, \tau) \to R^k$, we introduce notation for the right derivative at time $t \in [t_0, \tau)$

$$D^+q(t) \triangleq \lim_{\Delta t \to +0} \frac{1}{\Delta t}[q(t+\Delta t) - q(t)]$$

By a right-hand solution of the Cauchy problem

$$\ddot{q} = G(t,q,\dot{q}), q(t_0) = q_0, \dot{q}(t_0) = \dot{q}_0 \tag{3.1}$$

defined in $[t_0, \tau)$, we mean a continuous right differentiable function $(q(t), \dot{q}(t))$ satisfying the equations

$$D^+q(t) = \dot{q}(t), \quad D^+\dot{q}(t) = G(t,q(t),\dot{q}(t))$$

for all $t \in [t_0, \tau)$ (see [6]).

Any right-hand solution is also a Carathéodory solution (in the usual sense; see, for example, [6, 8]).

Theorem 3.1. Suppose that the assumptions of Section 1 concerning the continuity and differentiability of the functions in (1.1) hold and that a point $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ satisfying system (1.1) exists at which $|N_s| \neq 0, s \in (1, \ldots, k_*)\backslash(\mathcal{N}_0\backslash\mathcal{N}_1)$ and conditions (2.3) are satisfied. Then a number $\delta_0 > 0$ exists such that differential equations (1.1) are locally uniquely solvable for $\ddot{q}$ and can be reduced to the form (3.1) for $(t, q, \dot{q}) \in D_{\delta_0}$ (the Cauchy problem for these equations is formulated as in (3.1)), and a right-hand solution $(q(t), \dot{q}(t))$ of the Cauchy problem (3.1) in some interval $[t_0, \tau)$, $\tau > t_0$ exists which is a right-hand solution of the equations of dynamics (1.1); if conditions (1.2) are satisfied, it is also a right-hand solution of the equations of motion of a mechanical system with sliding friction.

Proof. The assertion that Eqs (1.1) are solvable and can be reduced to the form of (3.1) was established in Section 2. To prove the existence of a right-hand solution of problem (3.1), we consider the differential inclusion

$$\ddot{q} \in H(t,q,\dot{q}), \quad q(t_0) = q_0, \quad \dot{q}(t_0) = \dot{q}_0, \tag{3.2}$$

where $H(t, q, \dot{q})$ is the convex hull of the limit value of G at the point $(t, q, \dot{q})$, including the value $G(t, q, \dot{q})$.

The existence of a local Carathéodory solution $(q(t), \dot{q}(t))$ of problem (3.2) follows from [9, Theorem 1]. Now, using [8, p. 56, Theorem 1], we conclude that

$$\operatorname{Cont} \dot{q}(t) \subset H(t,q(t),\dot{q}(t)) \tag{3.3}$$

for all $t \in [t_0, \tau)$, where $\tau > t_0$ is some number and $\operatorname{Cont} \dot{q}(t)$ is the contingency of $\dot{q}(\cdot)$ at the point t (at $t = t_0$—the right contingency $C\dot{q}(t)$). Now, if $(t, q(t), \dot{q}(t))$ is a point of continuity of G, then $G(t, q(t), \dot{q}(t)) = H(t, q(t), \dot{q}(t))$ and it therefore follows from (3.3) that

$$\operatorname{Cont} \dot{q}(t) = \ddot{q}(t) = G(t,q(t),\dot{q}(t)) \tag{3.4}$$

But if $(t, q(t), \dot{q}(t))$ is a point of discontinuity of G then, by Proposition 2.1, all the limit values of G, distinct from the value of G at $(t, q(t), \dot{q}(t)$, are strictly separable from Γ_0 by a hyperplane $L = \{\upsilon \in R^k: \langle a, \upsilon \rangle = 0\}$. Since $G(t, q(t), \dot{q}(t)) \in \Gamma_0, \dot{q}(t) \in \Gamma_0, \dot{q}(t) \in L$ and $\dot{q}(t + h) \in \Gamma_0$ for small $h > 0$, it follows that $H(t, q(t)\dot{q}(t)) \cap \Gamma_0 = G(t, q(t), \dot{q}(t))$ and $C^+\dot{q}(t) \subset \Gamma_0$. We then deduce from (3.3) that

$$T^+\dot{q}(t) = D^+q(t) = G(t, q(t), \dot{q}(t))$$

This equality, together with (3.4), completes the proof that a right-hand solution of problem (3.1) exists.

By [2, Lemma 3.1], if (1.2) is valid, this solution will also be a (right-hand) solution of the equations of motion of mechanical systems with sliding friction as described in [2]. This completes the proof.

4. CONTINUABILITY OF RIGHT-HAND SOLUTIONS

The definitions of the continuation of a solution, a non-continuable solution and a right maximum interval of existence of a solution are understood in the usual sense (see, for example, [10]).

We shall say that a right-hand solution $(q_\omega(\cdot), \dot{q}_\omega(\cdot))$ of problem (1.1), defined in $[t_0, \tau)$ and not continuable to the right, tends to the boundary of a domain Ω, if, for any compact set $W \subset \Omega$, a time $t_w < \omega$ exists such that $(t, q_\omega(t), \dot{q}_\omega(t)) \notin W$ for all $t_w < t < \omega$. Under these conditions, if $(q_\omega(\cdot), \dot{q}_\omega(\cdot))$ is a continuation of some solution $(q(\cdot), \dot{q}(\cdot))$ of system (1.1), we shall say that $(q(\cdot), \dot{q}(\cdot))$ is continuable to the boundary of Ω.

Theorem 4.1. Suppose that the assumptions of Section 1 concerning the continuity and differentiability of the functions in (1.1) are valid, and that for all $(t, q, \dot{q}, \ddot{q}) \in \Omega \times R^k$ such that $|N_s(t, q, \dot{q}, \ddot{q})| \neq 0$ we have inequalities analogous to (2.3) but with the right-hand side multiplied by a certain quantity $L = L(t, q, \dot{q})$, $0 \leqslant L < 1$. Then, at every point $(t, q, \dot{q}) \in \Omega$, Eqs (1.1) are uniquely solvable for $\ddot{q}$ (they can be reduced to the form (3.1)) and, for any initial data $(t_0, q_0, \dot{q}_0) \in \mathscr{A} \triangleq \{t, q, \dot{q}) \in \Omega, |N(t, q, \dot{q}, G)| \neq 0, s \in \mathscr{N}_1(t, q, \dot{q}, G)\}$, a local right-hand solution of the Cauchy problem (1.1) exists.

Indeed, considering $F = F_1, \ldots, F_{k_*}$, where $F_1, \ldots, F_{k_*}$ are the functions occurring on the right-hand sides of (2.1), let us apply the principle of contractive mappings with respect to the variables $(\ddot{q}^1, \ldots, \ddot{q}^{k_*})$ for any fixed $(t, q, \dot{q}) \in \Omega$. It follows that Eqs (2.1) (and hence also (1.1)) are solvable for $\ddot{q}$ and this implies the validity of Proposition 2.1 for any $(t_0, q_0, \dot{q}_0) \in \mathscr{A}$. The existence of a local right-hand solution of Eqs (1.1) for initial data $(t_0, q_0, \dot{q}_0) \in \mathscr{A}$ is now proved as in Theorem 3.1.

Henceforth, we shall assume that all the conditions of Theorem 4.1 are satisfied, and consider only right-hand solutions $(q(\cdot), \dot{q}(\cdot))$ of problem (1.1) such that $(t, q(t), \dot{q}) \in \mathscr{A}$ for all t in the domain of existence of that solution.

Theorem 4.2. Every right-hand solution $(q(\cdot), \dot{q}(\cdot))$ of the system of differential equations (1.1) is either continuable to the boundary of the set Ω or a limit of the mapping $t \to (t, q(t), \dot{q}(t))$ as $t \to \omega - 0$ exists, equal to $(\omega, q_*\dot{q}_*) \notin \mathscr{A}$.

Proof. The fact that $(q(\cdot), \dot{q}(\cdot))$ is continuable to the right maximum interval of existence is established by standard arguments, using Zorn's lemma. If $(q(\cdot), q(\cdot))$ does not tend to the boundary of Ω, then, using the fact that G is locally bounded, it can be shown that a limit of the mapping $t \to (t, q(t), \dot{q}(t))$ as $t \to \omega - 0$ exists which clearly cannot belong to the set $\mathscr{A}$. This implies the assertion of the theorem.

5. POINTS OF RIGHT UNIQUENESS

A right-hand solution $(q(\cdot), \dot{q}(\cdot))$ of system (1.1) is said to be R-right-hand at a point t if the right derivative $D^+\dot{q}(\cdot)$ is right continuous at t.

Under the assumptions of Theorem 4.1, R-right-hand solutions have the following property. A solution of system (1.1) that is R-right-hand at a point t_0 is R-right-hand at every point of some interval $[t_0, \tau)$, $\tau > 0$. Hence we may view Theorem 3.1 as an existence theorem for local R-right-hand solutions.

The definition of right uniqueness may be found in [8].

Theorem 5.1. Suppose that a point $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ at which inequalities (2.3) hold, $|N_s(t_0, q_0, \dot{q}_0, \ddot{q}_0| \neq 0$ for $s \in (1, \ldots, k_*)\backslash\mathscr{N}_0(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ and $\mathscr{N}_1(t_0, q_0, \dot{q}_0, \ddot{q}_0) = \phi$, satisfies system (1.1). Suppose that in some neighbourhood of the point $(t_0, q_0, \dot{q}_0)$ the functions Q_s^A, g_s, f_s are continuously differentiable with respect to $(q, \dot{q})$, and in some neighbourhood of the point $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ the functions $|N_s|$ are continuously differentiable with respect to $(q, \dot{q}, \ddot{q})$ (for every fixed t). Then any R-right-hand solution at t_0 of problem (1.1) with initial data $q(t_0) = q_0$, $\dot{q}(t_0) = \dot{q}_0$ is right unique at that point.

Proof. Since $\mathcal{N}_1(t_0, q_0, \dot{q}_0, \ddot{q}_0) = \phi$, it follows that a solution that is R-right-hand at t_0 locally satisfies (1.3) with fixed structure of the right-hand side generated by the point $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$. Let F_s^0 $(s = 1, \ldots, k)$ denote the functions, defined in a neighbourhood of $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$. through which this structure is expressed. It follows from inequalities (2.3) that the equation $\ddot{q} = F^0(t, q, \dot{q}, \ddot{q})$ is solvable and gives a unique function $G(t, q, \dot{q})$. By our assumptions, the function F^0 is continuously differentiable with respect to $(q\ \dot{q}, \ddot{q})$. Then G^0 will be continuously differentiable with respect to $(q, \dot{q})$ for every fixed t in some neighbourhood of the point $(t_0, q_0, \dot{q}_0)$, and every R-right-hand solution of system (1.1) with initial data $q(t_0) = q_0\ \dot{q}(t_0) = \dot{q}_0$ will locally satisfy the equation

$$\ddot{q} = G^0(t, q, \dot{q}) \tag{5.1}$$

By standard methods of the theory of ordinary differential equations it can now be shown that t_0 is a point of right uniqueness of the solution of problem (5.1), thus proving the theorem.

It should be noted that R-right-hand solutions possessing the uniqueness property correspond to the intuitive meaning of the motion of a mechanical system with sliding friction.

6. EXAMPLE

Consider a two-dimensional system of two rigid bodies: (1) a piston **B** of mass m_1 moving with sliding friction in a rectangular tube Ox inclined at an angle $\alpha = \text{const}$ $(0 \leqslant \alpha \leqslant \pi/2)$ to the horizontal, and considered as a material point whose coordinate x is taken as q^1; (2) an absolutely rigid body of mass m_2 rotating with friction about a cylindrical hinge, mounted on the piston with its centre of mass **C** a distance r from the piston, and with moment of inertia J_c relative to the centre of mass. The resistance of the medium is ignored. The inclination β of **BC** to the downward normal to Ox is taken as q^2; f_1 and f_2 are, respectively, the coefficient of sliding friction of the piston and the friction coefficient in the hinge; $m = m_1 + m_2$.

The equations of motion of the system, in Lagrangian form, may be written as follows:

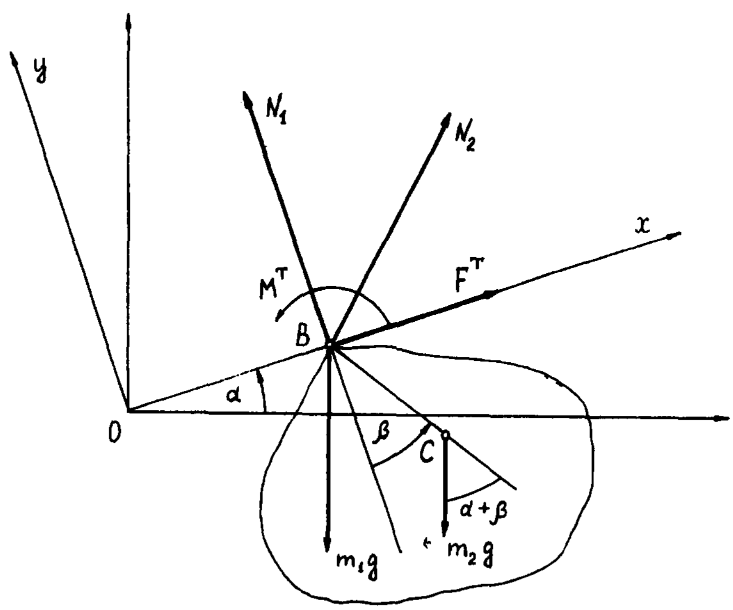

Fig. 1.

$$m\ddot{x} + m_2 r\cos\beta\,\ddot{\beta} = m_2 r\,\dot{\beta}^2\sin\beta - mg\sin\alpha + Q_1^T \tag{6.1}$$

$$m_2 r\cos\beta\,\ddot{x} + J\ddot{\beta} = -m_2 gr\sin(\alpha+\beta) + Q_2^T$$

where $Q_1^T, F^T, Q_2^T = M, J = J_C + m_2 r^2$. The generalized friction forces at relative equilibrium with respect to x and β may be written as follows:

$$Q_1^{T0} \triangleq m_2 r\ddot{\beta}\cos\beta - m_2 r\dot{\beta}^2\sin\beta + mg\sin\alpha \quad (\dot{x}=0,\ \ddot{x}=0)$$

$$Q_2^{T0} \triangleq m_2 r\ddot{x}\cos\beta + m_2 gr\ \sin(\alpha+\beta) \quad (\dot{\beta}=0,\ \ddot{\beta}=0)$$

The absolute values of the normal reactions, defined according to the rule described in [2], are

$$|N_1| = |m_2 r(\ddot{\beta}\sin\beta + \dot{\beta}\cos\beta) + mg\cos\alpha|$$

$$|N_2| = m_2[(\ddot{x} + r\ddot{\beta}^2\cos\beta - r\dot{\beta}^2\sin\beta + g\sin\alpha)^2 + (r\ddot{\beta}\sin\beta + r\dot{\beta}^2\cos\beta + g\cos\alpha)^2]^{1/2}$$

In the general case, the generalized friction forces are determined by the following equalities, $s = 1, 2$

$$Q_s^T = \begin{cases} Q_s^{T0}, & \text{if}\quad \dot{q}^s = 0,\ |Q_s^{T0}| \leqslant f_s|N_s|_{\ddot{q}^s=0} \\ f_s|N_s|\operatorname{sgn}Q_s^{T0}, & \text{if}\quad \dot{q}^s = 0,\ |Q_s^{T0}| > f_s|N_s|_{\ddot{q}^s=0} \\ -f_s|N_s|\operatorname{sgn}\dot{q}^s, & \text{if}\quad \dot{q}^s \neq 0 \end{cases}$$

Inequality (1.2) holds for $s = 1$, because $|N_1|$ is independent of $\ddot{x} = \ddot{q}^1$. If $s = 2$, then, differentiating $|N_2|$ with respect to $\ddot{\beta}$ (provided that $|N_2| \neq 0$), we find that a sufficient condition for (1.2) to be valid is that

$$f_2 m_2 r(|\cos\beta| + |\sin\beta|) < J \tag{6.2}$$

Since $\max_\beta(|\cos\beta| + |\sin\beta|) = \sqrt{2}$, it follows from (6.2) that

$$f_2 < J/(\sqrt{2}m_2 r) \tag{6.3}$$

In particular, if $J_C = 0$ (i.e. if the body is replaced by a point of mass m_2), then $f_2 < r/\sqrt{2}$. It is also obvious that inequalities (6.2) and (6.3) are valid if $f_2 = 0$.

The following conditions are sufficient for inequalities (2.3) to hold for Eqs (6.1)

$$\begin{aligned} &m_2 r(|\cos\beta| + f_1|\sin\beta|) < me_{12}/2 \\ &m_2(r|\cos\beta| + f_2) < Je_{21}/2 \\ &m_2 r f_2(|\cos\beta| + |\sin\beta|) < Je_{22}/2 \\ &(e_{12} = 1\text{м},\ e_{21} = 1\text{м}^{-1},\ e_{22} = 1) \end{aligned} \tag{6.4}$$

Clearly, the third inequality of (6.4) implies (6.2). Consider the set $\mathcal{A} = \{(q,\dot{q}): |N_s(q,\dot{q},G)| \neq 0, s \in \mathcal{N}_1(q,\dot{q},G)\}$.

1. Let $\mathcal{N}_1(q,\dot{q},G) = \{1,2\}$. Then $|Q_s^{T0}| = f_s|N_s|$, $\dot{q}^s = 0$, $\ddot{q}^s = 0$ $(s = 1, 2)$, and the condition $|N_2| = 0$ implies $g = 0$ or $m_2 = 0$, which is impossible. Hence it is always true that $(q,\dot{q}) \in \mathcal{A}$ if $\mathcal{N}_1 = \{1, 2\}$.

2. Let $\mathcal{N}_1(q,\dot{q},G) = \{2\}$. Then $|Q_2^{T0}| = f_2|N_2|$, $\dot{q}^2 = 0$, $\ddot{q}^2 = 0$ and the condition $|N_2| = 0$ means that

$$\ddot{x}\cos\beta + g\sin(\alpha+\beta) = 0,\quad \ddot{x} + g\sin\alpha = 0,\quad g\cos\alpha = 0 \tag{6.5}$$

The third equality of (6.5) gives $\alpha = \pi/2$, and it then follows from the second equality of (6.5) that $\ddot{x} = -g$. Since $|N_1|_{\beta=0,\ \dot{\beta}=0,\ \alpha=\pi/2} = 0$, $|Q_1^{T0}|_{\beta=0,\ \dot{\beta}=0,\ \alpha=\pi/2} > 0$, it follows that $\ddot{x} = -g$ satisfies Eqs (6.1) (and, obviously, the first equality of (6.5)). Consequently, if $\alpha = \pi/2$ and Eqs (6.1) are uniquely solvable for $\ddot{x}, \ddot{\beta}$ (which is guaranteed by inequalities (6.4)), the only points not contained in $\mathcal{A}$ are pairs $(q,\dot{q})$ such that $\dot{q}^2 = \dot{\beta} = 0$.

3. Let $\mathcal{N}_1(q,\dot{q},G) = \{1\}$. Then $\dot{q}^1 = 0$, $\ddot{q}^1 = 0$, $|Q_1^{T0}| = f_1|N_1|$ and the condition $|N_1| = 0$ implies that

$$\ddot{\beta}\cos\beta - \dot{\beta}^2\sin\beta + \xi\sin\alpha = 0,\quad \ddot{\beta}\sin\beta + \dot{\beta}^2\cos\beta + \xi\cos\alpha = 0 \tag{6.6}$$

$(\xi = mg/(m_2 r)$. Then $|N_2| = gm_1$.

Let $\dot{\beta} = 0$. Then $\ddot{\beta} \neq 0$ and, provided that $\cos\beta = 0$ and $\sin\beta = 0$, it follows from (6.6) and (6.1) that $\sin\alpha = 0$ or $\cos\alpha = 0$, respectively, and

$$Jm/(m_2r) = m_2r - f_2m_1 \tag{6.7}$$

Hence it follows that

$$J_c\frac{1}{m_2r} = -r + \frac{m_2}{m}r - f_2\frac{m_1}{m} < 0$$

which is impossible. If $\cos\beta \neq 0$, $\sin\beta \neq 0$, it follows from (6.6) that

$$\ddot{\beta} = -\xi\sin\alpha/\cos\beta = -\xi\cos\alpha/\sin\beta$$

Consequently, $\cos(\alpha + \beta) = 0$. Simple calculations show that in this case also (6.7) holds, and so the set $\mathcal{A}$ does not contain $(q, \dot{q})$ if $\dot{q}^1 = 0, \dot{q}^2 = 0, \mathcal{N}_1 = \{1\}$.

If $\dot{\beta} \neq 0$, then the values of $q^2 = \beta$, $\dot{q}^2 = \dot{\beta}$, for which $(q, \dot{q}) \in \mathcal{A}$ are found from the equations

$$RJ^{-1}\cos\beta - \dot{\beta}^2\sin\beta = -\xi\sin\alpha, \quad RJ^{-1}\sin\beta + \dot{\beta}^2\cos\beta = -\xi\cos\alpha \tag{6.8}$$

$$(R = -m_2gr\sin(\alpha+\beta) - f_2m_1g\,\mathrm{sgn}\,\dot{\beta})$$

Let β and $\dot{\beta}$ satisfy Eqs (6.8). Elementary algebra leads from (6.8) to the equalities

$$\dot{\beta}\sin(\alpha+\beta) = RJ^{-1}\cos(\alpha+\beta), \quad RJ^{-1} = -2\xi\sin(\alpha+\beta), \quad \dot{\beta}^2 = -\xi\cos(\alpha+\beta)$$

whence we obtain $R = 0$, $\sin(\alpha + \beta) = 0$. Then $f_2 = 0$ and $\cos(\alpha + \beta) = -1$. Consequently, the quantities

$$\beta = \pi - \alpha, \quad \dot{\beta}^2 = \xi$$

are solutions of Eqs (6.8) and it is necessary that $f_2 = 0$.

Combining the discussions of cases 1–3, we conclude that if Eqs (6.1) are uniquely solvable for $\ddot{q}$, then the only states not contained in A are $(q, \dot{q})$ such that

1. $\alpha = \pi/2, \dot{q}^2 = \dot{\beta} = 0$
2. $f_2 = 0, \dot{q}^1 = \dot{x} = 0, q^2 = \beta = \pi - \alpha, \dot{q}^2 = \dot{\beta} = \pm\sqrt{\xi}$.

Thus, if $0 \leqslant \alpha < \pi/2, f_2 > 0$ and inequalities (6.4) hold, then, by Theorem 4.1, a right-hand solution of Eqs (6.1) exists for any initial data $x_0, \dot{x}_1, \beta_0, \dot{\beta}_0$.

In conclusion, we note that Eqs (6.1) reduce to Painlevé's example ([1], see also [4]) if $\alpha = 0, f_2 = 0, m_1 = m_2 = 1$ and the body is replaced by a point **B**. In that case inequality (2.3) will hold if

$$f_1 < (1 + \sin^2\beta)/|\sin\beta\cos\beta|$$

which agrees with the condition for the resolution of the Painlevé paradoxes associated with "impossibility of non-uniqueness of motions". We have $(q, \dot{q}) \notin \mathcal{A}$ if $\dot{q}^1 = \dot{x} = 0, q^2 = \beta = \pi, \dot{q}^2 = \dot{\beta} = \pm\sqrt{(2g/r)}$.

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